Recursive Approximation of Analog Integrators

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Abstract — Recursive digital integrators (RDI) approximating the analog integrator frequency response are designed. Generic structures are considered and a set of designing rules is proposed. With this approximation it is possible to shape the introduced modulus and phase distortion in a desired way and it is possible to eliminate the trapezoidal integrator modulus distortion [1-3] while maintaining the phase error below a certain level. Direct comparison with classical first-order RDIs [1-3] is done. Design examples are provided. Application of RDIs to digital filter design is also considered.

1 Introduction

The main objective of recursive digital integrators design is to provide a good digital approximation of the analog integrator, \( I_A(j\Omega) \), which is characterized by constant phase and modulus slope for all frequencies.

\[
I_A(j\Omega) = \frac{1}{j\Omega}, \quad \Omega = \omega T_s
\]

\[
I_D(e^{j\Omega}) = I_A(j\Omega) \times EF(e^{j\Omega})
\]

The RDI is characterized by an error function, EF, which will be shaped in order to achieve the desired goals: \(|EF(e^{j\Omega})| \approx 1\) and \(\angle \{EF(e^{j\Omega})\} \approx 0\). Since it is impossible to design a perfect RDI \(EF(e^{j\Omega}) = 1\), depending on the application, the designer should shape the EF according to its objective: no phase error, no modulus error or a tentative of compromise between the two.

In the following error measures to evaluate RDI design are defined. Some 1st and 2nd-order RDI based on classical first-order digital integrators [1-3] are analyzed. A design method of \(M^{th}\)-order generic RDI with desired properties is proposed and characterized. Design examples are provided. The possibility of using the proposed RDI to obtain a good digital approximation of analog filters is addressed. Finally, conclusions are drawn.

2 Error Measures

In order to evaluate RDI design, error measures comparing the analog and digital integrator frequency responses are calculated. Frequency responses are evaluated over a discrete number, \(N\), of frequency points, \(\Omega_n\), in a frequency interval with chosen endpoints. In all design examples presented in next sections \(\Omega_n \in [0.01, 0.99]\pi\) and \(N=1000\). Modulus and phase root mean square (rms) errors are defined allowing both a quantitative comparison and the definition of objective functions to be minimized:

\[
E_M = \sqrt{\frac{1}{N} \sum_{n=0}^{N-1} [I_A(j\Omega_n) - I_D(e^{j\Omega_n})]^2}, \quad (3)
\]

\[
E_\phi = \sqrt{\frac{1}{N} \sum_{n=0}^{N-1} \angle \{I_A(j\Omega_n)\} - \angle \{I_D(e^{j\Omega_n})\}]^2. \quad (4)
\]

Taking the trapezoidal integrator, \(I_{BI}\), as a standard, the RDI modulus rms error reduction is also evaluated through: \(E_{R M} = 1 - E_M(I_D)/E_M(I_{BI})\). When a compromise solution between the two extremes is sought for, a mixed objective function may be used \((0 \leq \beta \leq 1)\):

\[
E_{M\phi} = \beta E_M(I_D) + (1 - \beta) \cdot \frac{E_\phi(I_D)}{\arg(I_{BI})/2}. \quad (5)
\]

Since a common measure to assess integrators is their quality factor, a loss factor is also calculated, \(L_D(e^{j\Omega}) = -\Im\{I_D(j\Omega)/\Re\{I_D(j\Omega)\}\}\), and a corresponding rms error evaluated:

\[
E_L = \sqrt{\frac{1}{N} \sum_{n=0}^{N-1} L_D(e^{j\Omega_n})^2} . \quad (6)
\]

3 Classical RDI

When the usage of recursive digital integrators is considered, the most common choices are first-order [1, 2]. The preferred RDI is usually the trapezoidal (bilinear) integrator because of its perfect phase characteristic and known frequency warping deformation. The main disadvantage of the trapezoidal integrator is the modulus distortion introduced for high frequencies [1-3] (see fig. 1).

Two other common 1st-order RDI with less modulus distortion are the backward and forward Euler integrators [1-3]. Although these integrators reproduce much better the analog integrator modulus (fig. 1), they introduce a significant amount of phase distortion (see fig. 2).

In order to try to overcome these integrators drawbacks, it is possible to design new 1st and 2nd-order RDI based on them. A 1st-order RDI allowing a very good modulus approximation is based on the backward and forward
Euler integrators weighted addition \((0 \leq \alpha \leq 1)\):

\[ I_{B+F}(e^{j\Omega}) = \frac{\alpha + (1 - \alpha)e^{-j\Omega}}{1 - e^{-j\Omega}}. \]  

When minimization of the modulus rms error, \(E_M\), is the objective \(\alpha = 0.8576837\) is found. When \(E_\phi\) is minimized \(I_{BI}\) is found. On the other hand when a compromise solution between these two extremes is sought for, through \(E_M\) minimization, a lower limit of \(\beta = 0.4\) is determined (before the solution converges to \(I_{BI}\)) and \(\alpha = 0.8160970\).

Independently from the objective criteria formulation although a very good approximation of the modulus characteristic is possible (see fig. 1), this integrator introduces significant phase error (see fig. 2).

The rms errors for this and all previously considered 1st-order RDI are presented in table 1.

\[
\begin{array}{|c|c|c|c|}
\hline
E_M(dB) & E_\phi(\alpha) & E_L & ER_M(\%) \\
\hline I_{BI} & 7.32 & 0 & 0 \\
I_{B, I_F} & 1.68 & 51.72 & 6.51 & 77.1 \\
I_{B+F (a)} & 0.279 & 46.27 & 4.66 & 96.2 \\
I_{B+F (b)} & 0.625 & 44.27 & 4.12 & 91.3 \\
I_{BF} & 10.55 & 0 & 0 & -44.1 \\
I_{BF+BI} & 2.58 & 0 & 0 & 64.8 \\
\hline
\end{array}
\]

Table 1: \(E_M, E_\phi, E_L\) and \(ER_M\) of \(I_{BI}, I_{B, I_F}, I_{B+F}\) ((a) \(\alpha = 0.8576837, \beta = 1\); (b) \(\alpha = 0.8160970, \beta = 0.4\)), \(I_{BF}\) and \(I_{BF+BI}\) \((\alpha = 0.9330598)\).

In order to try to guarantee a perfect phase characteristic a 2\(n^d\)-order RDI based on the backward and forward Euler integrators multiplication is defined as:

\[ I_{BF}(e^{j\Omega}) = \frac{2e^{-j\Omega}}{1 - e^{-j2\Omega}}, \]  

which corresponds to the lossless discrete integrator operating at a double frequency sampling [3]. Even though without phase error, this RDI presents a significant modulus distortion with a peak as \(\Omega \rightarrow \pi\) (see fig. 3 and tab. 1 for rms errors) which may not be shaped because all coefficients are set. Although by itself this integrator has no interest (\(I_{BI}\) is 1st-order and has much less modulus distortion), it allows the definition of a new 2\(n^d\)-order RDI without phase error but with the hypotheses of some modulus error shaping through an \(\alpha\) weight factor \((0 \leq \alpha \leq 1)\):

\[ I_{BF+BI}(e^{j\Omega}) = \alpha \cdot I_{BI}(e^{j\Omega}) + (1 - \alpha) \cdot I_{BF}(e^{j\Omega}) = \frac{\alpha + 2(1 - \alpha)e^{-j\Omega} + \alpha e^{-j2\Omega}}{2(1 - e^{-j\Omega})}. \]  

Nevertheless the degree of freedom introduced by \(\alpha\), it is impossible to eliminate the modulus peak as \(\Omega \rightarrow \pi\). Minimization of the \(E_M\) lead to \(\alpha = 0.9330598\) (see fig. 3 and tab. 1 for rms errors).

From this study based on the classical integrators it is possible to conclude that without more degrees of freedom it is impossible to obtain a very good digital approximation of the analog integrator modulus characteristic although a perfect phase characteristic may be obtained.

### 4 Design of \(M\)-order Generic RDI

A generic \(M^{th}\)-order recursive digital integrator (GRDI) is now considered and a specific set of design rules are established in order to provide better approximations of the analog integrator.

\[ I_D(e^{j\Omega}) = \frac{\sum_{m=0}^{M} a_m e^{-jm\Omega}}{\sum_{m=0}^{M} b_m e^{-jm\Omega}}. \]  

The digital integrator order, \(M\), is chosen according to the desired complexity. Without loss of generality \(b_0\) is set to 1. Coefficients \(a_m\) and \(b_m\) are calculated through
the application of restrictions to the GRDI and its EF. In this way, depending on the considered restrictions, it is possible to design a digital integrator whose frequency response (modulus and phase) approximates in a desired way the analog response; i.e., it is possible to shape the introduced module and phase distortion. Whenever there are coefficients which are not explicitly computed through the imposed restrictions, they must be computed through optimization methods as the weight \( \alpha \) was computed in last section.

The set of rules used to design the GRDI impose that the GRDI structure must be realizable and at \( \Omega = 0 \) its gain must be \( \infty \) and the EF should not introduce any error. When perfect phase is the objective the EF must be real. In this case, to prevent zeros in the modulus characteristic, it is necessary to impose that the EF remains positive for all frequencies. Finally, whenever the modulus presents undesired peaks as \( \Omega \rightarrow \pi \) a restriction imposing a finite final value is imposed. This set of rules is summarized in table 2 where each one is given a suggestive label. Depending on the way the restrictions are applied different levels of approximation are obtained.

<table>
<thead>
<tr>
<th>Restriction</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( GHI_nf )</td>
<td>( \lim_{\Omega \to 0} I_D(e^{j\Omega}) = \infty )</td>
</tr>
<tr>
<td>( ModP )</td>
<td>poles off ( I_D(e^{j\Omega}) \leq 1 )</td>
</tr>
<tr>
<td>( Lim0EF )</td>
<td>( \lim_{\Omega \to 0} EF(e^{j\Omega}) = 1 )</td>
</tr>
<tr>
<td>( ImEF )</td>
<td>( Im \ EF(e^{j\Omega}) = 0 )</td>
</tr>
<tr>
<td>( pEF )</td>
<td>( EF(e^{j\Omega}) &gt; 0 )</td>
</tr>
<tr>
<td>( LimxEF )</td>
<td>( \lim_{\Omega \to \pi} EF(e^{j\Omega}) \neq \infty )</td>
</tr>
</tbody>
</table>

Table 2: Set of GRDI design rules.

4.1 1\textsuperscript{st}-order GRDI

The simultaneous imposition of \( GHI_nf \), \( Lim0EF \) and \( ModP \) results in \( I_{GLOM}(e^{j\Omega}) = I_{B1+EF}(e^{j\Omega}) \) \( (a_0 = \alpha) \). When \( ImEF \) is applied the only solution obtained is \( I_{B1}(e^{j\Omega}) \). Then the 1\textsuperscript{st}-order GRDI set of solutions matches the analyzed 1\textsuperscript{st}-order classical RDI.

4.2 2\textsuperscript{nd}-order GRDI

The simultaneous application of \( GHI_nf \), \( Lim0EF \) and \( ModP \) results in two solutions. The \( I_{GLOM}(e^{j\Omega}) \) already considered and:

\[
I_{GLOM}(e^{j\Omega}) = \frac{a_0 + a_1 e^{-j\Omega} + \left(2 - a_0 - a_1 + b_1\right) e^{-2j\Omega}}{\left(1 - e^{-j\Omega}\right)\left(1 + (1 + b_1) e^{-j\Omega}\right)} \tag{11}
\]

with \(-2 < b_1 \leq 0\). When \( b_1 = 0 \) there is an amplitude peak as \( \Omega \rightarrow \pi \) and application of \( Lim\pi EF \) leads to \( I_{GLOM}(e^{j\Omega}) \). When \( ImEF \) and \( pEF \) are applied the only solution (besides \( I_{B1}(e^{j\Omega}) \)) is the already analyzed \( I_{B1+EF}(e^{j\Omega}) \) \( (\alpha = 2a_0) \) with \( a_0 < 1/2 \). As it was seen this GRDI has a final peak and its elimination leads to \( I_{B1}(e^{j\Omega}) \).

Figures 4 and 5 show the modulus (zoomed in) and phase characteristics of \( I_{GLOM}(e^{j\Omega}) \) for two different designs and the corresponding error measures are in table 3. In the first case \( E_M \) was minimized while in the second one \( E_{M\phi} \) was minimized with \( \beta = 0.4 \) (the same weight as before). In both cases the modulus may be con-

![Figure 3: Modulus characteristic of 2\textsuperscript{nd}-order GRDI.](image)

![Figure 4: Modulus characteristic of 2\textsuperscript{nd}-order GRDI.](image)
point of view, the case of \( \beta = 1 \) would be preferable because higher phase errors are all accumulated near \( \Omega = \pi \).

| \( I_{2GLOM} \) (a) | 0.030 | 22.33 | 1.11 | 99.6 |
| \( I_{2GLOM} \) (b) | 0.093 | 19.41 | 0.69 | 98.7 |

Table 3: \( E_M, E_\phi, E_L \) and \( ER_M \) of \( I_{2GLOM} \).

### 4.3 3\(^{rd}\)-order GRDI

The application of \( \text{GI}n\ f, \text{Lim0EF} \) and \( \text{ModPl} \) leads to several solutions besides the \( I_B(e^{j\Omega}) \), \( I_{GLOM}(e^{j\Omega}) \) and \( I_{2GLOM}(e^{j\Omega}) \). In this case different GRDI solutions are obtained depending on the type of poles apart from the one in \( z = 1 \). With two equal poles the GRDI is:

\[
I_{3GLOMPRd}(e^{j\Omega}) = \frac{4a_0 + 4a_1 e^{-j\Omega} + 4a_2 e^{-j2\Omega} + A_3 e^{-j3\Omega}}{(1 - e^{-j\Omega})[2 + (1 + b_1)e^{-j\Omega}]^2},
\]

with \( A_3 = 9 - 4(a_0 + a_1 + a_2) + b_1(6 + b_1) \) and \(-3 < b_1 < 1\). When \( b_1 = 1 \) there will be an amplitude peak at \( \Omega = \pi \). Application of \( \text{Lim0EF} \) reduces the solution to \( I_{3GLOM}(e^{j\Omega}) \).

With two different real poles:

\[
I_{3GLOMPRs}(e^{j\Omega}) = \frac{a_0 + a_1 e^{-j\Omega} + a_2 e^{-j2\Omega} + A_3 e^{-j3\Omega}}{(1 - e^{-j\Omega})[1 + (1 + b_1)e^{-j\Omega} + (1 + b_1 + b_2)e^{-j2\Omega}]},
\]

with \( A_3 = 3 - a_0 - a_1 - a_2 + 2b_1 + b_2 \). In this case the set of coefficient restrictions involves nonlinear constraints:

\[
\begin{align*}
-3 &< b_1 < 1 \\
2b_2 &\geq -1 \\
2b_2 &> -3 - 2b_1 \\
b_2 &< \left(\frac{1 - b_1}{4}\right)^2 - 1
\end{align*}
\]

A particular case occurs when \( b_2 = -1 \) where the final amplitude peak is present in this solution and an equivalent to \( I_{2GLOM} \) is found when \( \text{Lim} \)\( \phi \) \( E\)F is applied.

Finally, when a pair of complex poles is present \( I_{3GLOMPC}(e^{j\Omega}) = I_{3GLOMPRs}(e^{j\Omega}) \) is found but with a different set of coefficient restrictions:

\[
\left\{ \begin{array}{l}
-3 < b_1 < 1 \\
b_2 \leq -b_1 \\
b_2 > \left(\frac{1 - b_1}{4}\right)^2 - 1
\end{array} \right.
\]

In this case there are no particular values of \( b_1 \) and \( b_2 \) leading to amplitude peaks.

Contrarily to the previous GRDI orders, in this case, there are no solutions assuring simultaneously \( \text{ModPl} \), \( \text{LimEF} \) and \( pEF \). On the other hand when \( 4^{th} \)-order GRDI are designed there are solutions successfully achieving these three conditions but all present the final undesired amplitude peak. Its elimination always leads to previously analyzed GRDIs.

![Figure 6: Modulus characteristic of 3\(^{rd}\)-order GRDI.](image-url)

Figures 6 and 7 show several examples of 3\(^{rd}\)-order GRDI design and the corresponding error measures are in table 4. The double pole case with minimization of the \( E_M \) converged to a solution presenting high phase distortion because there is a pole-zero cancellation and although there are two poles, they are tied together. When \( E_{M\phi} \) was minimized (\( \beta = 0.4 \)) the found solution is identical to the one obtained with 2\(^{nd}\)-order GRDI. So the hypothesis of two real poles does not permit achieving better results than the ones of lower order GRDI.

When two different poles are considered there is the possibility of extending the range of solutions. In this case minimization of the \( E_M \) allowed a concentration of phase distortion near \( \Omega = \pi \) while maintaining a very low phase error for almost 70% of all the frequency range. When \( E_{M\phi} \) is minimized with \( \beta = 0.4 \) the solution converged to the particular case of \( b_2 = -1 \) where the error phase is negligible but there is amplitude distortion. The
limit case between these two solutions is also presented and was found for $\beta = 0.5$. Once more it is clear that the designer has to choose between phase and amplitude distortion. Finally, the case of conjugate complex poles is considered. This has proven to be the hardest case for coefficient optimization because coefficient solutions converged to the border approaching the real pole case. With $\beta = 0.25$ it was possible to reach a solution presenting a compromise between phase and modulus distortion.

$$\text{Table 4: } E_M, E_\phi, E_r \text{ and } ER_M \text{ of } 3^{rd}-\text{order GRDI } ((a)-\beta = 1, (b)-\beta = 0.4, (c)-\beta = 0.5, (d)-\beta = 0.25).$$

<table>
<thead>
<tr>
<th>$I_{3GL0MPrd}$ (a)</th>
<th>$E_M(dB)$</th>
<th>$E_\phi(%)$</th>
<th>$E_r$</th>
<th>$ER_M(%)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.030</td>
<td>44.87</td>
<td>3.37</td>
<td>99.6</td>
</tr>
<tr>
<td>$I_{3GL0MPbrd}$ (b)</td>
<td>0.093</td>
<td>19.41</td>
<td>0.687</td>
<td>98.7</td>
</tr>
<tr>
<td>$I_{3GL0MPbrs}$ (a)</td>
<td>0.018</td>
<td>23.84</td>
<td>1.42</td>
<td>99.8</td>
</tr>
<tr>
<td>$I_{3GL0MPbrs}$ (b)</td>
<td>2.58</td>
<td>0</td>
<td>0</td>
<td>64.8</td>
</tr>
<tr>
<td>$I_{3GL0MPbrs}$ (c)</td>
<td>2.11</td>
<td>2.72</td>
<td>0.048</td>
<td>71.1</td>
</tr>
<tr>
<td>$I_{3GL0MPcc}$ (d)</td>
<td>0.935</td>
<td>27.74</td>
<td>1.67</td>
<td>87.2</td>
</tr>
</tbody>
</table>

5 GRDI Application to Filters

The usual method of mapping between the S and Z domains, in order to design digital filters which accurately approximate the frequency response of an analog filter is the bilinear integrator because it assures stability and there is a perfect match between the $j\Omega$ axis and the unitary circle. GRDI may also be used to digital filter design although these two characteristics may not be generically assured. Its advantage is the possibility to correct phase distortion usually introduced by the bilinear integrator but at the cost of amplitude distortion. An illustrative example is presented in figure 8. A lowpass Chebychev $5^{th}$-order digital filter obtained using the $f_B$ and the $I_{3GL0MPbrs}$ is shown. With the GRDI it was possible to obtain $R_E = 92\%$ and $R_E = 27\%$. Inside the filter band it was possible to assure perfect phase match but at the cost of amplitude distortion.

6 Conclusion

The proposed GRDI design method to approximate the analog integrator frequency response allowed the shaping of the introduced amplitude and phase distortion. Solutions without phase error and solutions without amplitude error were found. It was then possible to shape the error in order to reach the opposite situation of the bilinear integrator were there is only amplitude distortion. Here GRDI coefficients were found through the minimization of the modulus or phase RMS errors or by means of a compromise solution trying to minimize both objectives. Since the minimization of the modulus and phase RMS errors does not occur for simultaneous coefficients it was found that solutions converge to one of the two minimums. So in order to really achieve a solution representing a compromise between the two types of distortion optimization methods should not be used and direct pole and zero position placement may be used. The higher the GRDI order the better approximations may be. The application of GRDI to filter design although possible may not be applied as a general rule because depending on the filter characteristics distortion and even instability may arise.

7 References